Lecture 16 : Continuity Theorem for Characteristic Functions

STAT205 *Lecturer: Jim Pitman* Scribe: *Cristine Pinto* <cristine@econ.berkeley.edu>

References: [1], section 2.3.

16.1 Review of the Inversion Formula

Recall that if X is a random variable, it's characteristic function is

$$\varphi_X(t) = \mathbb{E}\left[e^{itX}\right].$$

In the last lecture, we proved the Uniqueness Theorem for characteristic functions. We also learned the *inversion formula*: if $\int |\varphi_X(t)| < \infty$, then X has a bounded continuous density

$$f_X(x) = \frac{1}{2\pi}\int e^{-itx}\varphi_X(t)dt.$$

16.2 Continuity Theorem for Characteristic Functions

Suppose we have a sequence of distributions on the line $(\mathbb{P}_n)$ with characteristic functions

$$\varphi_n(t) = \int e^{itx} \cdot \mathbb{P}_n(dx) = \mathbb{E}\left[e^{itX_n}\right]$$

for $X_n \sim \mathbb{P}_n$. We want to be able to tell that $\mathbb{P}_n$ converges in distribution to some limit on $\mathbb{R}$ by looking at $\varphi_X(t)$. Recall that if $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ then

$$\int f d\mathbb{P}_n \longrightarrow \int f d\mathbb{P} \text{ for every bounded continuous function } f.$$

So in particular for $f(x) = e^{itx}$, we get $\varphi_n(t) \longrightarrow \varphi(t)$ as $n \longrightarrow \infty$ where $\varphi(t) = \int e^{itx}\mathbb{P}(dx)$, the characteristic function of $\mathbb{P}$.

Consider the converse. Suppose we have a sequence $\mathbb{P}_n$ with $\varphi_n(t)$ and $\varphi_n(t) \longrightarrow \varphi(t)$ as $n \longrightarrow \infty$ for some function $\varphi(t)$. Without imposing further assumptions, we cannot conclude that $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ where $\mathbb{P}$ has a continuous characteristic function $\varphi(t)$.

Example 16.1 *Let* $\mathbb{P}_n = N(0,n)$*, so that* $\varphi_n(t) = e^{-\frac{1}{2}nt^2}$*,* $t \in \mathbb{R}$*. Notice that* $\varphi_n(t) \longrightarrow \mathbf{1}(t=0)$*, but*

$$\mathbb{P}_n \xrightarrow{v} \frac{1}{2}\delta_{-\infty} + \frac{1}{2}\delta_{+\infty}.$$

Recall the idea of *tightness*: $(\mathbb{P}_n)$ is tight if:

$$\lim_{x\to\infty} \sup_n \mathbb{P}_n(-x,x)^c = 0.$$

We present the first (easy) version of the continuity theorem for characteristic functions.

Theorem 16.2 *Let* $\mathbb{P}_n$ *be probability measures on* $\mathbb{R}$ *with c.f.* φ_n*. If:*

1. $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ *exists for every* $t \in \mathbb{R}$*; and*
2. $\mathbb{P}_n$ *is tight,*

then $\mathbb{P}_n \longrightarrow \mathbb{P}$ *where* $\mathbb{P}$ *is a probability measure on* $\mathbb{R}$ *with c.f.* φ*.*

Proof: By Helly's selection theorem, to prove that there exists a $\mathbb{P}$ such that $\mathbb{P}_n \longrightarrow \mathbb{P}$, it is enough to show that there exists a $\mathbb{P}$ such that every subsequence of $(\mathbb{P}_n)$ has a further subsequence which converges to $\mathbb{P}$.

To find a suitable $\mathbb{P}$, recall the general theorem: Let $\mathcal{C}$ be a collection of bounded continuous functions which is determining. If:

1. $\lim_{n\to\infty} \int f d\mathbb{P}_n$ exists for all $f \in \mathcal{C}$; and
2. $(\mathbb{P}_n)$ is tight,

then:

$$\mathbb{P}_n \xrightarrow{d} \mathbb{P} \text{ where } \int f d\mathbb{P} = \lim_{n\to\infty} \int f d\mathbb{P}_n,\ \forall f \in \mathcal{C}$$

Apply this general theorem to

$$\mathcal{C} = \left\{ f \text{ of the form } f(x) = \sin(tx),\ t \in \mathbb{R} \text{ or } f(x) = \cos(tx),\ t \in \mathbb{R} \right\}.$$

$\mathcal{C}$ is determining by the uniqueness theorem for c.f.'s. ■

This form of the continuity theorem is adequate for most applications, such as the CLT. Usually in the CLT we try to show:

$$Z_n = \frac{S_n}{\sqrt{\mathbb{E}(S_n^2)}} \xrightarrow{d} N(0,1)$$

In this case, $\mathbb{P}_n$ is the distribution of Z_n with $\mathbb{E}(Z_n) = 0$. Clearly, $\mathbb{P}_n$ is tight:

$$\mathbb{P}_n(-x, x)^c = \mathbb{P}(|Z_n| > x) \leq \frac{\mathbb{E}(Z_n^2)}{x^2} = \frac{1}{x^2},$$

which decreases to 0 as $x \uparrow \infty$.

So, if we continue to assume condition 1 of theorem 16.2, condition 2 implies that there exists some $\mathbb{P}$ with c.f. φ such that:

(2a) φ is the characteristic function of *some* distribution. This also implies

(2b) φ is continuous as a function of t. This, in turn, implies

(2c) $t \longrightarrow \varphi(t)$ is continuous at $t = 0$.

Paul Lévy found that with condition 1, (2b) is *equivalent* to condition 2 of theorem 16.2.

Theorem 16.3 *(Lévy Continuity Theorem for c.f.'s): Given $\mathbb{P}_n$ with c.f. φ_n, if:*

1. *$lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ exists for all $t \in \mathbb{R}$; and*
2. *$t \longrightarrow \varphi(t)$ is continuous at $t = 0$,*

then

$$\mathbb{P}_n \xrightarrow{d} \mathbb{P} \textit{ with } \int e^{itx}\, \mathbb{P}(dx) = \varphi(t).$$

Proof: In the proof, we just need to show that continuity of φ at 0 implies $(\mathbb{P}_n)$ is tight. For that, it's most convincing to use a genuine bound ([1], p. 98):

$$\frac{1}{u} \cdot \int_u^{-u} [1 - \varphi_n(t)]dt \geq \mathbb{P}_n \left(\frac{-2}{u}, \frac{2}{u} \right)^c$$

Recall that $\varphi_n(t) \longrightarrow \varphi(t)$ as $n \longrightarrow \infty$ and $\varphi_n(t)$ is continuous at $t = 0$, $\varphi(0) = 1$. As $n \longrightarrow \infty$, then,

$$\frac{1}{u} \int_{-u}^{u} [1 - \varphi_n(t)]dt \longrightarrow \frac{1}{u} \int_{-u}^{u} [1 - \varphi(t)]dt,$$

and as $u \longrightarrow 0$,

$$\frac{1}{u} \int_{-u}^{u} [1 - \varphi(t)]dt \longrightarrow 0.$$

Fix $\epsilon > 0$ and choose u small enough so that $\frac{1}{u}\int_{-u}^{u}[1-\varphi(t)]dt < \epsilon$. Choose N large enough that:

$$\frac{1}{u}\int_{-u}^{u}|1-\varphi_n(t)|dt < 2\epsilon \text{ for } n \geq N.$$

Now we have

$$\mathbb{P}_n\left(-\frac{2}{u},\frac{2}{u}\right)^c \leq 2\epsilon \text{ for all } n \geq N,$$

and hence $\lim_{x\to\infty}\sup_n \mathbb{P}_n(-x,x)^c = 0$ as desired. ■

16.3 Exercises

Exercise 16.4 (Extra credit problem) *Suppose a sequence* $(\mathbb{P}_n)$ *of probability measure on* $\mathbb{R}$ *such that* $\lim_{n\to\infty}\int f d\mathbb{P}_n$ *exists and* $\in \mathbb{R}$ *for every bounded continuous* f*. Then (you check): there exists a unique probability measure* $\mathbb{P}$ *such that* $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$*. Consequently:*

$$\int f d\mathbb{P} = \lim_{n\to\infty}\int f d\mathbb{P}_n.$$

Exercise 16.5 *What happens if we replace* $\mathbb{R}$ *in exercise 16.4 by* $\mathbb{R}^n$ *or a generic metric space?*

Example 16.6 (Related to the exercises above) *Consider:*

$$\mathcal{C} = \{f : f \text{ is bounded, continuous, and has a compact support}\}$$

(Note that if f *has compact support, then* $f(x) = 0$ *for* $|x| > B$ *for some* $B \geq 0$*.) Check that* $\mathcal{C}$ *is a determining class.*

Consider

$$\mathcal{C}_0 = \{f : f(0) = 0 \text{ and } f \text{ is continuous with compact support}\};$$

$\mathcal{C}_0$ *is also a determining class.*

Let

$$\mathbb{P}_n = \begin{cases} \delta_n & \text{if } n \text{ is even} \\ \delta_0 & \text{if } n \text{ is odd} \end{cases}$$

In this case, $\int f d\mathbb{P}_n \longrightarrow \int f d\delta_0$ *for all* $f \in \mathcal{C}$*, but* $\mathbb{P}_n \not\twoheadrightarrow \mathbb{P}_0$*.*

The moral of this example is that to prove that $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ *for some probability* $\mathbb{P}$ *on* $\mathbb{R}$*, it is not enough to just show that* $\int f d\mathbb{P}_n \longrightarrow \int f d\mathbb{P}$ *for all* f *in a determining class.*

References

[1] Richard Durrett. *Probability: theory and examples, 3rd edition.* Thomson Brooks/Cole, 2005.